AN INVESTIGATION ON FUNCTIONAL MODELS FOR FERTILIZER RESPONSE SURFACES

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Judicious use of fertilizers form one of the most important means of stepping up agricultural production. For efficient fertilizer use, it is necessary to have information on the optimum doses and combinations of fertilizers under different soil climatic conditions. This information is generally obtained from the results of field experiments testing combinations of nutrients such as nitrogen, phosphorus and potash at different levels. The dose yield relationship is established by fitting a suitable mathematical function to the yield data. The fitted response function is then studied in respect of isoquants, isoclines, marginal substitution rates between nutrients etc., and the optimum doses are estimated from the function. Choice of suitable functional models for describing the dose-yield relationship is, therefore, one of the important steps in fertilizer use research. In the present study, an attempt has been made to examine the suitability of different mathematical models of multi-variate response surfaces for describing fertilizer-yield relationship using the data of experiments available in India. As nitrogen and phosphorus are the two nutrients which have been generally tried and which have shown response, the investigation is confined to these two nutrients only. All the experiments considered pertain to rice or wheat crop, since suitable experiments on other crops were extremely few.

Types of Response Functions Considered

Since only two nutrients are taken, the general form of response considered is $y=\phi(x, z)$. The particular types of functions considered are described below. With sufficiently high levels of fertilizer application, diminishing returns take place generally and so only functions exhibiting diminishing returns are considered here.

(*i*) *The Mitscherlich-Baule Function* :

$$y=a\{1-e^{-c(x+b)}\}\{1-e^{-k(z+d)}\}$$

The ratio of the yields y and y', corresponding to two levels of a nutrient x, for a fixed level of z, is independent of the level at which z is taken. Hence this function accounts for interaction in the sense that interaction arises from the failure of the difference between y and y' to remain constant over different levels of z. The constants 'b' and 'd' measure the equivalent amounts of nutrients available to the crop in the unmanured soil. The constants 'c' and 'k' measure the importance of the factors to the crop. These are always positive. The constant 'a' measures the maximum yield.

(*ii*) *The generalised Cobb-Douglas function* :

$$y=a(x+b)^c\,(z+d)^e$$

In this case too, as the ratio y/y', of yields corresponding to two levels of one factor is independent of the constant 'a', it is independent of the influence of other factors when the value of one factor changes in intensity. This function cannot account for declining yield with increased doses of nutrients. As the doses of fertilizers increase, the yield increases. The constants 'b' and 'd' measure the equivalent amounts of nutrients in the unmanured soil. The constants 'c' and 'e' measure the importance of the fertilizers to the crop and they are called coefficients of elasticity. With diminishing rates of response to the factors, 'c' and 'e' should be positive and numerically less than one.

(*iii*) *Maskell Resistance Formula.*

Balmukand (1928) not satisfied with the Mitscherlich function when applied to field data, critically examined another yield dose relationship suggested by Maskell. Maskell's formula may be termed by electrical analogy as 'Resistance Formula' in the form

$$\frac{1}{y}=a+\frac{b}{x+c}+\frac{d}{z+e}\ .$$

This expression like Mitscherlich's assumes that one factor acts independently of the other, but fixes the differences of reciprocals of yields $\left(\frac{1}{y}-\frac{1}{y'}\right)$ as constants. The constants 'c' and 'e' represent the amount of available nutrients to the crop in the unmanured soil, and 'b' and 'd' measure the importance of the nutrients to the crop. This surface too does not account for declining yields with increased doses of fertilizers. When the responses to the factors are positive 'b' and 'd' will be positive and 'a' will always be positive since $\frac{1}{a}$, gives the maximum yield.

(*iv*) *Quadratic response surface.*

$$y=a+bx+cx^2+dz+ez^2+fzx.$$

where '*b*' and '*d*' are linear effects and '*c*' and '*e*' are the quadratic effects of the factors *x* and *z* respectively. '*f*' represents the interaction effect of the two factors, *i.e.*, the extent to which the response to the combination of the two is different from the sum of the responses to the individual nutrients. This coefficient corresponds to linear by linear interaction of *x* and *z* in factorial analysis. The constant '*a*' being the yield, when the levels of *x* and *z* applied are zero each, will be positive. If there is a positive response to *x* and *z* factors, following the law of diminishing returns, '*b*' and '*d*' will be positive whereas '*c*' and '*e*' will be negative. This will in general be the position with fertilizers. But '*f*' may be positive or negative according as the interaction between the factors is positive or negative. When there is no interaction between the factors, then '*f*' will be zero subject to experimental error.

(*v*) *Quadratic square-root transformation formula.*

$$y=a+b\sqrt{x}+cx+d\sqrt{z}+ez+f\sqrt{zx}.$$

This function is arrived by substituting $\sqrt{x}$ for *x*, and $\sqrt{z}$ for *z*, in the expression for the quadratic surface. The coefficient '*f*' accounts for interaction between the factors. This function will be preferable to the quadratic surface when the ratios of responses are relatively low.

DATA INCLUDED

The data included here are those of fertilizer experiments conducted in India. A fairly exhaustive search of published data of the required type based on experiments conducted in India was made for this purpose. The publications referred to were mainly 'Indian Journal of Agricultural Science', 'Bulletin of Manuring of Rice in India,' and the reports on Agricultural Stations in Madras State. In addition, the data of a large series of fertilizer trials conducted under a joint Indo-American Programme were also included. Only factorial experiments, whether complete or incomplete, with at least three levels of each factor can provide the basic data for fitting response functions of the type considered here. The number of experiments considered and other details of the experiments are given below : [Table (*a*)].

TABLE (*a*)

Crop	*No. of Centres*	*Location of Centres*	*Nature of treatments*	*No. of experiments*	*Period covered*	*Source of data*
Rice	11	T.C.M. Agronomic Trial Centres	3×3 factorial with N and P	30	1953-54 to 1955-56	T.C.M. trails (I.A.R.S.)
Wheat	8	—do—	—do—	17	—do—	—do—
Rice	1	Suri farm West Bengal	—do—	13	1948-49 to 1955-56	I.C.A.R. report
Rice	2	Suri and Berhampore West Bengal	3×4 factorial with N and P	12	—do—	I.C.A.R. report
Rice	1	Gaya (Bihar)	4×4 factorial with N and P	1	1936-37	Manuring of rice in India I.C.A.R. bulletin.
Rice	1	Chandkuri	—do—	1	—do—	—do—
Rice	1	Chinsura	5×3 factorial with N and P	18	1948-49 to 1955-56	I.C.A.R. report.
Rice	1	Berhampore	—do—	15	1949-50 to 1955-56	—do—

The quadratic response function was fitted to the data of all experiments, while the other types of response functions were fitted only to those data which exhibited some interaction between factors as *prima facie* these functions cannot be suitable in the absence of interaction.

METHOD OF FITTING

The fitting of quadratic response functions is done by the method of multiple regression, where the method of least squares is employed. The fitting of quadratic surface $y=a+bx+cx^2+dz+ez^2+fzx$ in the general case where the two factors x, and z, are tried at m, and n, levels which are equispaced, can be conveniently done by the method of orthogonal polynomials. The estimates of the constants and their variances are given in Appendix *A*.

In the case of quadratic functions whether in the variables or in their square roots discussed above, the response surfaces are linear functions of the constants and the fitting by least square method is therefore relatively easy. But in the case of the other three surfaces, *viz.*, (1) Resistance formula, (2) Cobb-Douglas function and (3) Mitscherlich-Baule functions, the formulas are non-linear functions of the constants. Hence the fitting of the functions by least square method is difficult and involves heavy computations. The general method of fitting in these cases is to get a set of provisional values for the constants and improve them by successive corrections. These corrections are obtained by a set of equations in each case. To reduce the computations it is essential to have a fairly good choice of initial values.

The procedure of obtaining the inital values and their correction as well as the S. Es. of the constants mainly corresponds to that of Bhai Balmukand (1928), who has given the method in the case of the Resistance formula. With slight modification, the same method can be used to fit Cobb-Douglas and Mitscherlich surfaces.

RESULTS

The analysis of variance of the fitted surface is of the form :—

Source of variation	*Degrees of freedom*	*Mean square*
Fitted surface	$k-1$	B_r
Deviations from fitted surface	$n-k$	B_d
Experimental error	d	E

The adequacy of the fit is determined by the lack of significance of the deviation from fitted surface tested against the experimental error. In Table 1, the number of experiments which indicated significant deviation compared to experimental error from the fitted regression function is given :—

TBBLE 1

Adequacy of the Quadratic Response function

Crop	No. of Centres	Total no. of experiments	No. of experiments with significant deviations from fitted surface	Number of experiments with different percentage variation accounted by the fitted surface			
				$\geqslant 90$	$\geqslant 80$ <90	$\geqslant 60$ <80	<60
Paddy	14	92	16	45	22	17	8
Wheat	9	15	0	11	2	2	0

The deviations from the fitted quadratic function were not significant in 85 percent of the experiments, indicating thereby the high suitability of this functional form for generally fitting fertilizer response function. Of the 16 experiments which showed significant departure from fitted surface on paddy 13 were at three experimental stations of Chinsura, Berhampore and Suri in West Bengal. All these experiments were repeated on the same site in different years and cannot therefore be considered as independent evidence. The proportion of total variation removed by the fitted regression surface is also shown in the above table. More than 80 per cent of the variation is accounted by the fitted regression function in 80 out of 107 experiments. In none of the experiments the quadratic components were positive and significant.

In six experiments, where there was evidence of interaction between factors other response surfaces were also fitted. The analysis of variance for testing the significance of the fitted surfaces and the deviations from the fit is given in Table 3. The equations of the fitted suafaces and the standard errors of the fitted constants are given below in Table 2.

The percentage standard errors of the constants in Mitscherlich function are appreciably small.

Relative fit : All the five functions fit adequately to the data considered. This fact can be seen from the analysis of variance given in Table 3.

In some of the trials, the deviations from the fitted surface is significant, showing the adequacy of the fit. The estimate of the experimental error of mean yields could not be obtained for Halwad, Chandukuri and Gaya Centres. As the experiments were conducted under not very dissimilar conditions, the extent of experimental error noted in other experiments provide some indication of the same for the experiments where the estimates of error are not available. On this basis, the deviation from fitted surfaces does not appear large relative to the order of experimental error at these centres also. Square root and Resistance formulae fit slightly better in the case of Tirurkuppam, Chandkuri and Gaya. In the remaining Centres, all the surfaces have shown the same degree of fit. In the quadratic and square root functions, there are five independent constants, whereas in the other three functions they are only four independent constants. These three functions have removed as much variation as the quadratic and square root functions, except at Gaya and Tirurkuppam.

TABLE 2

The fitted response function with standard error of constants

Centre	*Model*	*Fitted function*	*Standard Errors of the Constants*					
			a	*b*	*c*	*d*	*e* or *k*	*f*
Bhagwai (Wheat)	Quadratic	$y=6.26971+0.13813x-0.00251x^2+0.58504z-0.00832z^2+0.00464xz$		0·0621	0·00122	0·0621	0·00122	0·00030
	Square root	$y=6.63653+1.4460\sqrt{x}+0.00958x+1.91976\sqrt{z}-0.05633z+0.17297\sqrt{xz}$		0·3911	0·0606	0·3911	0·0606	0·0327
	Resistance	$\frac{1}{y}=0.01722+\frac{0.60056}{x+19.10744}+\frac{0.57974}{z+5.79420}$	0·0021	1·3386	10·488	0·1150	3·968	
	Cobb-Douglas	$y=4.26978(x+5.92592)^{0.18153}(z+0.93462)^{0.28421}$	1·6065	6·6340	0·2064	0·3720	0·2299	
	Mitscherlich	$y=29.25569\{1-e^{-0.03820(x+26.37535)}\}\{1-e^{-0.05141(z+7.26508)}\}$	1·4779	2·0140	0·0039	0·4646	0·00131	
Bhagwai (Paddy)	Quadratic	$y=14.17279+0.42042x-0.00517x^2+0.32908z-0.00399z^2+0.00334xz$		0·1322	0.0030	0·1322	0·0030	0·0022
	Square root	$y=14.35229+1.26769\sqrt{x}+0.01109x+0.90892\sqrt{z}-0.02348z+0.08610\sqrt{xz}$		0·9718	0·1473	0·9718	0·1473	0·0257
	Resistance	$\frac{1}{y}=0.01541+\frac{0.60434}{x+20.67770}+\frac{0.24818}{z+12.60452}$	0·0 023	0·5504	14.259	0·2919	11·7870	
	Cobb-Douglas	$y=7.80997(x+6.0420)^{0.22389}(z+3.96040)^{0.15410}$	8·6029	9·092	0·3405	17·8506	0·6373	
	Mitscherlich	$y=38.98608\{1-e^{-0.03586(x+23.50623)}\}\{1-e^{-0.04522(z+23.25500)}\}$	7·1554	3·280	0·0075	2·7928	0·0071	
Tirurkuppam (Paddy)	Quadratic	$y=7.29476+0.08675x-0.00057x^2+0.27540z-0.00377z^2+0.001524xz$		0·0067	0·00002	0·0383	0·00003	0·00016
	Square root	$y=8.65312+0.07645\sqrt{x}+0.00420x+1.17353\sqrt{z}-0.13190z+0.13932\sqrt{xz}$		0·2345	0·0197	0·3448	0·0425	0·0195
	Resistance	$\frac{1}{y}=0.02927+\frac{1.80284}{x+31.08933}+\frac{0.22110}{z+4.03737}$	0·0016	0·4521	7·1373	0·4839	2·1714	
	Cobb-Douglas	$y=3.93526(x+14.70588)^{0.29418}(z+0.00429)^{0.07140}$	1·6643	9·9720	0·1769	0·1885	0·0074	
	Mitscherlich	$y=24.70807\{1-e^{-0.01383(x+46.37866)}\}\{1-e^{-0.07402(z+9.43565)}\}$	2·0569	5·3478	0·0021	0·7036	0·0040	

Halwad (Wheat)	Quadratic	$y=2\cdot14501+0\cdot05225x-0\cdot00028x^2+0\cdot12742z-0\cdot00178z^2+0\cdot00301zx$
	Square root	$y=2\cdot31037-0\cdot29686\sqrt{x}+0\cdot08822x+0\cdot16114\sqrt{z}+0\cdot03097z+0\cdot10584\sqrt{xz}$
Halwad (Wheat)	Resistance	$\frac{1}{y}=-0\cdot07546+\frac{9\cdot58910}{x+37\cdot10598}+\frac{2\cdot08912}{z+9\cdot12174}$
	Cobb-Douglas	$y=0\cdot02245(x+36\cdot63004)^{1\cdot07913}(z+3\cdot65296)^{0\cdot39113}$
	Mitscherlich	$y=35\cdot53165\{1-e^{-0\cdot0000005(x+32\cdot23623)}\}\{1-e^{-0\cdot03246(z+11\cdot31307)}\}$
Gaya (Paddy)	Quadratic	$y=9\cdot91793+0\cdot16181x-0\cdot00153x^2+0\cdot24225z-0\cdot00366z^2+0\cdot00412xz$
	Square root	$y=11\cdot39524-0\cdot55239\sqrt{x}+0\cdot11598x+0\cdot87200\sqrt{z}-0\cdot10956z+0\cdot25184\sqrt{xz}$
	Resistance	$\frac{1}{y}=0\cdot00740+\frac{2\cdot04632}{x+32\cdot33910}+\frac{0\cdot28702}{z+6\cdot88196}$
	Cobb-Douglas	$y=1\cdot29549(x+26\cdot60270)^{0\cdot61814}(z+0\cdot03124)^{0\cdot08134}$
	Mitscherlich	$y=43\cdot61065\{1-e^{-0\cdot01795(x+21\cdot01154)}\}\{1-e^{-0\cdot10949(z+7\cdot15863)}\}$
Chandkuri (Paddy)	Quadratic	$y=10\cdot20425+0\cdot08210x-0\cdot00088x^2+0\cdot38139z-0\cdot00472z^2+0\cdot00167xz$
	Square root	$y=10\cdot57172-0\cdot00823\sqrt{x}+0\cdot01830x+2\cdot07045\sqrt{z}-0\cdot17323z+1\cdot0669\sqrt{xz}$
	Resistance	$\frac{1}{y}=0\cdot03030+\frac{0\cdot90062}{x+34\cdot46026}+\frac{0\cdot15934}{z+3\cdot54226}$
	Cobb-Douglas	$y=10\cdot31940(x+6\cdot90256)^{0\cdot13589}(z+0\cdot00070)^{0\cdot05612}$
	Mitscherlich	$y=23\cdot35566\{1-e^{-0\cdot03576(x+36\cdot49907)}\}\{1-e^{-0\cdot10765(z+7\cdot12351)}\}$

Note : (1) The equations are considered as given in the following for convenience to show the standard errors of the constant in each model in the above table.

1. Quadratic $y=a+bx+cx^2+dz+ez^2+fzx$
2. Square root $y=a+b\sqrt{x}+cx+d\sqrt{z}+ez+f\sqrt{zx}$
3. Resistance $\frac{1}{y}=a+\frac{b}{x+c}+\frac{d}{z+e}$
4. Cobb-Douglas $y=a(x+b)^c\,(z+d)^e$
5. Mitscherlich $y=a\{1-e^{-c(x+b)}\}\{1-e^{-k(z+d)}\}$

Note : (2) As usual y denotes the yield (mds/acre) whereas x and z represent N and P_2O_5 (lb/acre) in all cases.

TABLE 3.

Analysis of variance for testing goodness of fit of the different response functions for different centres.

Type of fitted function	Source of variation	*Bhagwai (wheat)* D.F.	M.S.	% variation removed by fitted surface	*Bhagwai (paddy)* D.F.	M.S.	% variation removed by fitted surface	*Halwad* D.F.	M.S.	% variation removed by fitted surface	*Chandkuri* D.F.	M.S.	% variation removed by fitted surface	*Gaya* D.F.	M.S.	% variation removed by fitted surface	*Tirürkuppam* D.F.	M.S.	% variation removed by fitted surface
Quadratic	Fitted function	5	72·949	100	5	58·430	97	5	12·847	99	5	59·388	95	5	115·823	94	5	75·188	95
	Dev. from fitted function	3	0·153		3	3·346		3	0·209		10	1·645		10	3·401		9	2·087	
Square root	Fitted function	5	72·965	100	5	58·398	97	5	12·688	98	5	52·006	99	5	117·914	96	5	76·176	96
	Dev. from fitted function	3	0.127		3	3·399		3	0·474		10	0·337		10	2·356		9	1·538	
Resistance	Fitted function	4	91·264	100	4	72·599	96	4	16·120	99	4	77·391	99	4	148·701	96	4	95·530	97
	Dev. from fitted function	4	0·149		4	2·948		4	0·097		11	0·348		11	1·996		10	1·260	
Cobb-Doublas	Fitted function	4	90·749	99	4	73·028	97	4	15·989	99	4	76·636	98	4	142·649	93	4	92·610	94
	Dev. from fitted function	4	0·553		4	2·518		4	0·225		11	0·623		11	3·867		10	2·429	
Mitscherlich	Fitted function	4	90·749	99	4	73·028	97	4	15·986	99	4	76·412	98	4	142·495	93	4	92·851	94
	Dev. from fitted function	4	0·553		4	2·518		4	0·225		11	0·705		11	3·922		10	2·332	
Expr. error		24	0·4775		24	2·948											14	0·881	

This fact is in favour of the functions with only four constants. The resistance formula has given uniformly best fit. It can be seen that the sum of squares accounted by Cobb-Douglas and Mitscherlich formulae, in all the centres are nearly equal.

COMPARISON OF OPTIMUM DOSES OBTAINED FROM DIFFERENT SURFACES.

The primary use of the fitted response function is to determine optimum nutrient combination for given cost price situation. The optimum doses are obtained when the marginal value of the produce is equal to the marginal cost and are given by the solution of the equations

$$p\frac{\partial\varphi}{\partial x} = q$$

$$p\frac{\partial\varphi}{\partial z} = r$$

where q and r, are the prices per unit of x and z respectively and p, is the price per unit of produce. The formulae for optimum doses with different surfaces are given in appendix B.

The optimum dose, say, x_0 and z_0 are functions of fitted constants. The variances of x_0 and z_0 can, therefore, be estimated approximately by using the formula ;—

$$\sigma_f^2 \simeq \left(\frac{\partial f}{\partial a}\right)^2 \sigma_a^2 + \left(\frac{\partial f}{\partial b}\right)^2 \sigma_b^2 + \left(\frac{\partial f}{\partial c}\right)^2 \sigma_c^2 + \ldots\ldots\ldots\ldots +$$

$$2\left(\frac{\partial f}{\partial a}\right)\left(\frac{\partial f}{\partial b}\right)\sigma_{ab} + 2\left(\frac{\partial f}{\partial a}\right)\left(\frac{\partial f}{\partial c}\right)\sigma_{ac} + \ldots\ldots\ldots\ldots$$

where σ_f^2 is the variance of a function $\sigma_f^2 f(a,b,c\ldots)$ of the fitted constants, σ_a^2, σ_b^2 etc. are the variances and σ_{ab}, σ_{ac} etc. the covariances of the estimated constants a, b, c etc. $\frac{\partial}{\partial a}$, $\frac{\partial}{\partial b}$ etc. are the partial derivatives of, f, with respect to a, and so on.

Using these formulae, the optimum doses and their corresponding standard errors were obtained for the different surfaces. These are given in Table 4.

The optimum values for all centres except Tirurkuppam and Chandkur differ very much from surface to surface. But at the two centres, Chandkuri and Tirurkuppam, the estimated optimum values from each of the five models are relatively close to one another. In the case of Bhagwai the optimum values estimated by quadratic and Mitscherlich are not far removed from the highest dose of 40 lb/acre tried.

TABLE 4

Optimum doses of N and P (with their standard errors)

Centre	*Bhagwai (wheat)*			*Bhagwai (paddy)*			*Chandkuri (wheat)*			*Tirurkuppam (paddy)*		
Type of the Surface	*Opt. dose*		*Opt. yield*	*Opt. dose*		*Opt. yield*	*Opt. dose*		*Opt. yield*	*Opt. dose*		*Opt. yield*
	N lb/ acre	*P_2O_5lb/ acre*	*md/ acre*	*N lb/ acre*	*P_2O_5lb/ acre*	*md/ acre*	*N lb/ acre*	*P_2O_5lb/ acre*	*md/ acre*	*N lb/ acre*	*P_2O_5lb/ acre*	*md/ acre*
Quadratic	59·83 (28·50)	48·71 (10·48)	27·84	43·67 (17·71)	46·14 (23·83)	34·07	41·94	40·98	22·68	63·27 (8·23)	41·02 (5·30)	19·41
Square root							24·01	30 25	20·01	26·86 (9·09)	23·71 (5·20)	15·27
Resistance							37·41	29·92	21·01	53·15 (24 02)	28·65 (27·29)	17·41
Cobb-Douglas							33·93	17·53	19·62	47·82 (34·30)	18·62 (3·80)	16·37
Mitscherlich	48·71 (3·51)	56·88 (3·54)	26·88	56·27 (12·38)	49·20 (9·94)	36·37	29·38	27·13	20·43	58 03 (31·00)	32 48 (3·21)	18·03

Note: The cost of 1 lb of N and P are Rs. 0·7701 and Re.·06278 respectively and Rs. 10 and 12 per maund of paddy and wheat.

The optimum values obtained from the Resistance and Cobb-Douglas formulae for both wheat and paddy at Bhagwai are far above the doses tried and hence are not presented in the table. The optimum values for Gaya centre by the different models are all extrapolated values. The square root formula failed to give optimum values in four out of six centres. For the data of Halwad the optimum with any of the surfaces could not be determined. It can be observed from the same table that the optimum values obtained by quadratic and Mitscherlich functions in all the centres are appreciably nearer.

The estimated S. Es. of the optimum doses are very high in each model particularly when extrapolation is involved. Again the S.Es. of the optimum doses in the cases of quadratic and Mitscherlich functions are small when compared to other surfaces. No generalization of those findings are possible with the limited numbers of experiments over which these different surfaces could be fitted.

Comparison between estimated values of the nutrients available in the manured soil by different surfaces.

It was mentioned in earlier section, that the three response functions such as Resistance formula, Cobb-Douglas and Mitscherlich-Baule function, estimate the value of the nutrients available to the plant in the unmanured soil. Estimates of the available nutrients from different surfaces are given in Table 5 and 6.

TABLE 5

The estimated amount of Nitrogen (lb/acre) available to the plant in the unmanured soil.

Centre	Type of the response function					
	Resistance		Mitscherlich		Cobb-Douglas	
		S.E.		S.E.		S.E.
1. Bhagwai (Wheat)	19·11	10·49	26·38	2·01	5·92	6·63
2. Bhagwai (Paddy)	20·68	14·26	23·51	3·28	6·04	9·09
3. Chandkuri	34·46	—	36·50	—	6·90	—
4. Gaya	32·34	—	21·01	—	26·60	—
5. Halwad	37·11	—	32·24	—	36.63	—
6. Tirurkuppam	31·09	7·13	46·38	5·34	14·71	9·97

TABLE 6

The estimated values of Phosphorus (lb/acre) available to the plant in the unmanured soil.

Centre	Type of the response function					
	Resistance		Mitscherlich		Cobb-Douglas	
		S.E.		S.E.		S.E.
1. Bhagwai (Wheat)	5·79	3·97	7·27	0·46	0·93	0·37
2. Bhagwai (Paddy)	12·60	11·79	23·25	2·79	3·96	17·85
3. Chandkuri	3·54	—	7·12	—	0·00	—
4. Gaya	6 88	—	7·16	—	0·03	—
5. Halwad	9·12	—	11·31	—	3·65	—
6. Tirurkuppam	4·04	2·17	9·44	0·70	0·00	0·19

In the six centres, the estimates of the nutrients available in the soil, obtained by Resistance formula and Mitscherlich-Baule function are in agreement with each other. The estimates obtained by the Cobb-Douglas functions are generally very small when compared with the estimates obtained by the other two functions.

Discussion and Conclusion

The results discussed in the foregoing sections, based on the data of a large number of trials, clearly show the adequacy of the quadratic response surface for describing yield dose relations with fertilizers. Even in the few cases where there was a significant departure from a quadratic fitted surface, the other specialized functions fitted here were not superior. Fitting a quadratic response surface is simple as only linear estimation is involved and the usual techniques of analysis of variance and test of significance can be immediately applied with this surface. This function also takes account of declining yields in part of the range of doses tried. The standard errors of estimated optimum doses based on quadratic surface are also not appreciably higher than those given by Mitscherlich function, this function giving generally the lowest standard errors. The Cobb-Douglas function appears to be the least suitable. The fact that the coefficients of the quadratic surface can be identified with the factorial effects is an added advantage of this surface.

APPENDIX A

Estimates of constants and the variances of a two variable quadratic response function fitted to the data obtained from a $m \times n$ complete factorial with equal spacing of levels of factors, No of replications $= r$.

Response functions $Y = a + bx + cx^2 + dz + ez^2 + fxz$

	m & n odd		*m even; n odd*		*m and n even*	
Constant	*estimator*	*variance of the estimate*	*estimator*	*variance*	*estimator*	*variance*
a	$\hat{a} = \frac{G}{mn} - \frac{m^2-1}{12} x_q - \frac{n^2-1}{12} z_q$	$\frac{\sigma^2}{mnr}\left[1 + \frac{5}{4}\left(\frac{m^2-1}{m^2-4} + \frac{n^2-1}{n^2-4}\right)\right]$	$\frac{G}{mn} - \frac{1}{2} x_l + \frac{1}{4} x_q - \frac{m^2-1}{12} x_q - \frac{n^2-1}{12} z_q$	$\frac{\sigma^2}{mnr}\left[1 + \frac{138}{m^2-1} + \frac{45}{4(m^2-1)(m^2-4)}\right]$	$\frac{G}{mn} - \frac{1}{2}\left(x_l + z_l\right) + \frac{1}{4}\left(x_q + z_q + x_l z_l\right) - \frac{m^2-1}{12} x_q - \frac{n^2-1}{12} z_q$	$\frac{\sigma^2}{mnr}\left[1 + 3\left(\frac{1}{m^2-1} - \frac{1}{n^2-1}\right) + \frac{45}{4}\left\{\frac{81}{(m^2-1)(m^2-4)} + \frac{1}{(n^2-1)(n^2-4)}\right\} + \frac{9}{(m^2-1)(n^2-1)} + \frac{5}{4}\left(\frac{m^2-1}{m^2-4} + \frac{n^2-1}{n^2-4}\right)\right]$
b	$\hat{b} = x_l$	$\frac{12\sigma^2}{mn\,(m^2-1)\,r}$	$x_l - x_q$	$\frac{12\sigma^2}{mn(m^2-1)r}\left(1 + \frac{15}{m^2-4}\right)$	$x_l - x_q - \frac{1}{2} x_l z_l$	$\frac{12\sigma^2}{mn(m^2-1)r}\left[1 + \frac{15}{m^2-4} + \frac{3}{n^2-1}\right]$
c	$\hat{c} = x_q$	$\frac{180\,\sigma^2}{mn(m^2-1)(m^2-4)r}$	x_q	$\frac{180\sigma^2}{mn(m^2-1)(m^2-4)r}$	x_q	$\frac{180\sigma^2}{mn(m^2-1)(m^2-4)r}$
d	$\hat{d} = z_l$	$\frac{12\sigma^2}{mn(n^2-1)r}$	$z_l - \frac{1}{2} x_l z_l$	$\frac{12\sigma^2}{mn\,(n^2-1)r}\left(1 + \frac{3}{m^2-1}\right)$	$z_l - z_q - \frac{1}{2} x_l z_l$	$\frac{12\sigma^2}{mn(n^2-1)r}\left[1 + \frac{15}{n^2-4} + \frac{3}{m^2-1}\right]$

e	$\hat{e}=z_q$	$\frac{180\,\sigma^2}{mn(n^2-1)(n^2-4)r}$	z_q	$\frac{180\sigma^2}{mn(n^2-1)(n^2-4)r}$	z_q	$\frac{180\sigma^2}{mn(m^2-1)(n^2-4)r}$
f	$\hat{f}=x_l z_l$	$\frac{144\,\sigma^2}{mn(m^2-1)(n^2-1)r}$	$x_l\, z_l$	$\frac{144\sigma^2}{mn(m^2-1)(n^2-1)r}$	$x_l\, z_l$	$\frac{144\sigma^2}{mn(m^2-1)(n^2-1)r}$
Covariances between estimated constants is zero.			Covariance $\left(\hat{b},\hat{c}\right)=\frac{180\sigma^2}{mn(m^2-1)(m^2-4)r}$ Covariance $\left(\hat{d},\hat{f}\right)=\frac{72\sigma^2}{mn(m^2-1)(n^2-1)}$ Other covariances are zero.		$cov.\left[\hat{b}\,\hat{c}\right]=\frac{-180\sigma^2}{mn(m^2-1)(m^2-4)r}$; $cov.\left[\hat{b}\,\hat{d}\right]=\frac{-36\sigma^2}{mn(m^2-1)(n^2-1)r}$ $cov.\left[\hat{b}\,\hat{f}\right]=\frac{-72\sigma^2}{mn(m^2-1)(n^2-1)r}$ $cov.\left[\hat{d}\,\hat{e}\right]=\frac{-180\sigma^2}{mn(m^2-1)(m^2-4)r}$ $cov.\left[\hat{d}\,\hat{f}\right]=\frac{-72\sigma^2}{mn(m^2-1)(n^2-1)r}$	

Note:—x_l and z_l are the linear and x_q, z_q are the quadratic components of the corresponding marginal means of factors x and z (yates, 1937)

G=grand total of the mean yields

σ^2=experimental error.

Ref: Yates, F. The design and analysis of factorial experiments. Imp. Bur. Soil Sci. Tech. Comm 35, 1937.

APPENDIX B

Optimum combinations given by different formulae.

(1) Quadratic
$$x_0 = \frac{f(d - r/p) - 2e(b - q/p)}{4ce - f^2}$$

$$z_0 = \frac{f(b - q/p) - 2c(d - r/p)}{4ce - f^2}.$$

(2) Square root
$$x_0 = \left[\frac{fd - 2b(e - r/p)}{4(c - q/p)(e - r/p) - f^2}\right]^2$$

$$z_0 = \left[\frac{fb - 2d(c - q/p)}{4(c - q/p)(e - r/p) - f^2}\right]^2.$$

(3) Resistance
$$x_0 = \frac{\sqrt{b}(1 - m\sqrt{b} - n\sqrt{d})}{ma} - c$$

$$z_0 = \frac{\sqrt{d}(1 - m\sqrt{b} - n\sqrt{d})}{ma} - e$$

where $m^2 = q/p$

and $n^2 = r/p$

(4) Cobb-Douglas
$$x_0 = \left[\frac{1}{a}\left(\frac{c}{m}\right)^{1-e}\left(\frac{e}{n}\right)^{e}\right]^{\frac{1}{1-c-e}} - b$$

$$z_0 = \left[\frac{1}{a}\left(\frac{e}{n}\right)^{1-c}\left(\frac{c}{m}\right)^{c}\right]^{\frac{1}{1-c-e}} - d.$$

where $m = q/p$

and $n = r/p.$

(5) Mitscherlich
$$e^{-cx_0} = \frac{-(B - A - bd) \pm \sqrt{(B - A - bd)^2 - 4Abd}}{2d}$$

where $A = \frac{m}{ac}$ and $B = \frac{n}{ak}.$

Similarly z_0 can be also estimated.

SUMMARY

The data of 107 factorial experiments testing variations in the levels of nitrogen and phosphorous on wheat and paddy in India were examined to study the form of the appropriate response function. The five response functions Mitscherlich, Resistance formula, Cobbs-Douglas, quadratic and square root formula were considered. It was found that due to the general absence of interaction between the factors only in six experiments all the different surfaces could be fitted. In the rest of the cases, the quadratic surface could be fitted. This surface gave an adequate fit in more than 85% of the experiments. The fitted surface removed more than 80% of the yield variation in about 70% of the experiments.

Among the other functions, resistance formula gave uniformly better fit when interaction was present. Estimates of the nutrients available in the soil were made using Mitscherlich, Resistance and Cobb-Douglas functions. It was found that the estimates by Mitscherlich and Resistance formula were close to each other, but those by Cobb-Douglas function were extremely low.

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